4.3-4.4 linear ODEs

Wednesday, March 3, 2021 4:00 AM

1st - order ODEs

$$\frac{dx}{dt} + a_1(t)\,x = g(t) \qquad x(t_0) = x_0$$

Recall : Integrating factor $I(t) = \exp\left(\int_{t_0}^{t} a_1(\tau)\,d\tau \right)$

$$\dot{I}(t) = I(t)\,a_1(t)$$

$$\underbrace{I(t)\frac{dx(t)}{dt} + a_1(t) I(t)\, x(t)}_{\text{exact derivative}} = I(t)\, g(t)$$

$$\frac{d\left(x(t)\, I(t)\right)}{dt} = I(t)\,\dot{x}(t) + x(t)\,\dot{I}(t)$$
$$= I(t)\,\dot{x}(t) + x(t)\, I(t)\, a_1(t).$$

$$\Rightarrow \frac{d\left(x(t)\, I(t)\right)}{dt} = I(t)\, g(t).$$

$$\int_{t_0}^{t} \qquad \Rightarrow \left[x(t) I(t) - x(t_0) I(t_0) \right] = \int_{t_0}^{t} I(\tau)\, g(\tau)\, d\tau$$

$$I(t_0) = \exp\left(\int_{t_0}^{t_0} a_1(\tau)\,d\tau \right) = \exp(0) = 1$$

$$\Rightarrow x(t) I(t) = x(t_0) + \int_{t_0}^{t} I(\tau)\, g(\tau)\, d\tau$$

$$\Rightarrow x(t) = \frac{1}{I(t)}\left[x_0 + \int_{t_0}^{t} I(\tau)\, g(\tau)\, d\tau \right]$$

$$\Rightarrow x(t) = e^{-\int_{t_0}^{t} a_1(\tau)\,d\tau} \left[x_0 + \int_{t_0}^{t} e^{\int_{t_0}^{\tau} a_1(u)\,du}\, g(\tau)\, d\tau \right]$$

If $a_1(t) \equiv a_1$ a constant, and $t_0 = 0$, then

$$x(t) = e^{-a_1 t}\left[x_0 + \int_0^t e^{a_1 \tau}\, g(\tau)\, d\tau \right]$$

If $g \equiv 0$, then

$$x(t) = x_0\, e^{-a_1 t} \qquad \text{(homogeneous solution)}$$

Ex. $\frac{dx}{dt} - \frac{x}{t} = te^{3t}$, $x(1) = 2$

Let $I(t) = e^{-\int \frac{1}{t} dt} = e^{-\ln t} = \frac{1}{t}$

$$\frac{1}{t}\frac{dx}{dt} - \frac{x}{t^2} = e^{3t}$$

$$\Rightarrow \frac{d(xt^{-1})}{dt} = e^{3t}$$

$$\frac{x}{t} = C + \frac{e^{3t}}{3}$$

$$x = \underbrace{Ct}_{\text{homog. solution}} + \underbrace{\frac{te^{3t}}{3}}_{\text{particular solution}}$$

general solution

Solve IVP; plug in $x(1) = 2$

$$2 = C + \frac{e^3}{3}$$

$$C = 2 - \frac{e^3}{3}$$

$$\Rightarrow \underbrace{x(t) = \left(2 - \frac{e^3}{3}\right)t + \frac{te^{3t}}{3}}_{\text{Solution to IVP.}}$$

Recall: A general solution to an nth-order linear nonhomogeneous ODE is the sum of the homogeneous solution and the particular solution.

$$x_{gen} = x_h + x_p.$$

The general homog solution $x_h(t) = \sum_{i=1}^{n} c_i \phi_i(t)$, where $\phi_i(t)$ are n linearly ind. solutions to the homog equation.

fundamental set of solutions

Constant coefficients

Let $\overset{(n)}{x} + a_1 \overset{(n-1)}{x} + \cdots + a_{n-1}\dot{x} + a_n x = 0$

Characteristic poly. $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$

The roots of $P(\lambda)$ (sometimes called eigenvalues of the ODE) give us a set of lin. ind. solutions

$t^{\beta} e^{\lambda t}$, where $\beta = \{0, 1, \ldots, m(\lambda) - 1\}$ and $m(\lambda)$ is the multiplicity of λ as a root of $P(\lambda)$.

Ex. $P(\lambda) = \lambda^4 - 1$, char. poly. of $\frac{d^4x}{dt^4} - x = 0$

$(\lambda^2-1)(\lambda^2+1) = 0$

$\lambda = \pm 1, \pm i$, so $e^t, e^{-t}, e^{it}, e^{-it}$ are solutions.

$$x(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{it} + c_4 e^{-it}$$

Ex. $P(\lambda) = \lambda^2 - 2\lambda + 1$, char. poly. of $\ddot{x} - 2\dot{x} + x = 0$

$(\lambda - 1)^2 = 0$

$\Rightarrow \lambda = 1$, $m(\lambda) = 2$, so e^t and te^t are solutions

$$x(t) = c_1 e^t + c_2 t e^t$$

Recall: $\text{span}\{e^{(a+bi)t}, e^{(a-bi)t}\} = \text{span}\{e^{at}\cos(bt), e^{at}\sin(bt)\}$

Ex. For $x^{(4)} - x = 0$

$$x(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

Thm 4.3 If all the roots of the char. poly. $P(\lambda)$ have negative real parts, then given any solution $x(t)$ of the corresponding homog. ODE, $\exists\, M, b > 0$ s.t.

$$|x(t)| \leq M e^{-bt} \quad \text{for } t > 0$$

and $$\lim_{t\to\infty} |x(t)| = 0.$$

proof. let $\lambda_1, \dots, \lambda_r$ be the r distinct eigenvalues, $m(\lambda)$ be the multiplicity.

Then $x(t) = \sum_{i=1}^{r} \sum_{j=0}^{m(\lambda_i)-1} c_{ij} t^j e^{\lambda_i t}$. Let $b < \min_{i=1,\dots,r} |\text{Re}\, \lambda_i|$

Note $|t^j e^{\lambda_i t} e^{bt}| = |t^j e^{(\text{Re}\,\lambda_i + b)t}|$

And $\lim_{t\to\infty} \left| t^j e^{(\mathrm{Re}\,\lambda_i + b)t} \right| = 0$ (because $\lim_{t\to\infty} t^j e^{-kt} = 0$ for all $k > 0$)

$\Rightarrow \exists\, M_i > 0$ s.t. $\left| t^j e^{\lambda_i t} e^{bt} \right| < M_i$

$\Leftrightarrow \quad \left| t^j e^{\lambda_i t} \right| < M_i e^{-bt}$

Let $M_0 = \max\{c_{ij}\}$ and let $M = M_0 \sum_{i=1}^{r} \sum_{j=1}^{m(\lambda_i)-1} M_i$

Then $|x(t)| \leq M e^{-bt}$ $\boxed{}$